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Discrete-Time Model Reduction in Limited Frequency Ranges

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A mathematical formulation for model reduction of discrete-time systems such that the reduced-order model represents the system in a particular frequency range is discussed. The algorithm transforms the full-order system into balanced coordinates using frequency-weighted discrete controllability and observability grammians. In this form a criterion is derived to guide truncation of states based on their contribution to the frequency range of interest. Minimization of the criterion is accomplished without need for numerical optimization. Balancing requires the computation of discrete frequency-weighted grammians. Closed-form solutions for the computation of frequency-weighted grammians are developed. Numerical examples are discussed to demonstrate the algorithm.

Introduction

WHEN designing controllers for large dimensional systems, the first problem one must face is the model reduction. There have been numerous papers dealing with the problem. They all consider two major approaches. The first approach uses optimality conditions in conjunction with optimization algorithms to perform an exhaustive search for an optimal reduced-order model. The second approach uses special coordinate transformations to transform the system into a so-called balanced form. In this form the states are easily arranged in order of importance. The ordering is based on the state contribution to either the pulse response for the deterministic formulation or the response to white noise for the stochastic counterpart. The second approach yields a suboptimal solution, but with a significant reduction in computational time. The work in Ref. 1, which addresses the first approach, presents the initial formulation of the optimal model reduction problem, including necessary and sufficient conditions for an optimal solution to exist. This work was later extended and refined and a comparison of the various approaches was presented.[2] Solutions in both cases are optimal in the sense that they minimize the response error between the reduced- and full-order model. Because of the nonlinear optimization procedure, solutions using these approaches tend to be computationally intensive. A suboptimal solution to the model reduction problem is initially discussed in Ref. 3. A heuristic argument is given to justify truncation of certain states, but later a formal connection with the optimal reduction procedure is clearly established.[4] A similar procedure, known as component cost analysis, is presented in Ref. 5, and the connection with Ref. 4 is pointed out in Ref. 6. All of the suboptimal approaches rely on special transformations to minimize the coupling between states that are to be truncated and those to be retained. Near-optimum conditions for model reduction in balanced and modal coordinates are presented in Ref. 7. At the same time, a formulation for model reduction in limited time and frequency ranges was proposed in Ref. 8. The work discussed in this paper is an extension of the suboptimal model reduction solution for particular frequency ranges[8] to discrete time systems. The objective is to deal with discrete-time systems directly without need for conversion to continuous time before model reduction is performed.

The outline of the paper is as follows. First, the truncation error criterion is defined in terms of pulse responses. Second, the error criterion is transformed to frequency domain and expressed in terms of the controllability grammian. Third, a brief review is presented on how to use balanced coordinates for model reduction. Fourth, closed-form solutions for the discrete frequency-weighted grammians are obtained for use in balancing the system according to frequency. Finally, a numerical example is discussed to illustrate the algorithm.

Problem Statement

The model reduction problem addresses the question of how to reduce the number of states from the equations of motion by eliminating those contributing least to the total system response. The system equations for an nth order discrete time system are given by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \tag{1}$$

or by rearranging the order of the states one can write Eq. (1) in partition form

$$\begin{aligned} \begin{Bmatrix} x_r(k+1) \\ x_t(k+1) \end{Bmatrix} &= \begin{bmatrix} A_{rr} & A_{rt} \\ A_{tr} & A_{tt} \end{bmatrix} \begin{Bmatrix} x_r(k) \\ x_t(k) \end{Bmatrix} + \begin{Bmatrix} B_r \\ B_t \end{Bmatrix} u(k) \\ y(k) &= [C_r \quad C_t] \begin{Bmatrix} x_r(k) \\ x_t(k) \end{Bmatrix} \end{aligned} \tag{2}$$

where subscript r refers to states to be retained and t to states to be truncated. The objective function defined in terms of the error in the system response due to truncation is given by

$$J = \sum_{\tau=0}^{p-1} [y(\tau) - y_r(\tau)]^T [y(\tau) - y_r(\tau)] = \sum_{\tau=0}^{p-1} y_t^T(\tau) y_t(\tau) \tag{3}$$

The truncation error is accumulated over p sample points. The response of the system in Eq. (1) is easily propagated from time $k=0$ to any given sample time using

$$x(k) = A^k x(0) + \sum_{i=0}^{k-1} A^i B u(k-i-1) \tag{4}$$

To examine the response of two systems (the full- and reduced-order system), one can compare their corresponding pulse responses. Assuming the system is initially at rest, a

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pulse is applied to each of the inputs one at a time. From Eq. (4) the response due to a pulse at the ith input is

$$h^i(k) = A^{k-1} b_i \tag{5}$$

where b_i is the ith column of the B matrix. Collecting all $h^i(k)$ for q inputs, one can write the matrix of pulse responses as

$$\bar{X}(k) = [h^1(k)\ h^2(k)\ \cdots\ h^q(k)] = A^{k-1}B \tag{6}$$

Since the states are partitioned as in Eq. (2), the output pulse response matrix for $k > 0$ is given by

$$\begin{aligned}\bar{Y}(k) &= \bar{Y}_r(k) + \bar{Y}_t(k)\\ &= [C_r\ \ C_t]\bar{X}(k)\\ &= C_r\bar{X}_r(k) + C_t\bar{X}_t(k)\end{aligned} \tag{7}$$

The error performance measure in Eq. (3), based on pulse responses, can now be written using Eq. (7)

$$J = \sum_{t=1}^{p} \mathrm{tr}\{\bar{Y}_t(\tau)\bar{Y}_t^T(\tau)\} \tag{8}$$

where tr{ } refers to the trace of the matrix. Using Eq. (7) in Eq. (8) and trace properties, one may write the index as

$$J = \sum_{\tau=1}^{p} \mathrm{tr}\{C_t^T C_t \bar{X}_t(\tau)\bar{X}_t^T(\tau)\} \tag{9}$$

The discrete controllability grammian is

$$W_c(p) = \sum_{\tau=0}^{p-1} A^\tau BB^T(A^T)^\tau = \sum_{\tau=0}^{p-1} \bar{X}(\tau+1)\bar{X}^T(\tau+1) \tag{10}$$

where the second equality is a consequence of Eq. (6). The grammian is a real symmetric non-negative definite matrix. The argument p is used to stress the fact that it is a function of the number of sample points. Using r for states to be retained and t for states to be truncated, the grammian written in partition form is

$$W_c(p) = \begin{bmatrix} w_{rr} & w_{rt} \\ w_{tr}^T & w_{tt} \end{bmatrix} \tag{11}$$

where

$$w_{tt} = \sum_{\tau=0}^{p-1} \bar{X}_t(\tau+1)\bar{X}_t^T(\tau+1) \tag{12}$$

By substituting Eq. (12) into Eq. (9), one may express the truncation criterion in terms of the controllability grammian as

$$J = \mathrm{tr}\{C_t C_t^T w_{tt}\} \tag{13}$$

This criterion gives a measure of the truncation error when neglecting certain states and the system has been excited using pulses. The criterion can be shown to be identical if the performance measure is taken as the expected value of the truncated states output using white noise sequences as input to the system.

Truncation Criterion in the Frequency Domain

Truncation of states based on Eq. (3) or Eq. (13) minimizes the square error difference between the truncated and original system over a time window of p samples. The resulting truncated model will contain information over a broad frequency spectrum. A common practice for the experienced control engineer is to restrict the control actions to a certain frequency range. The range is often determined by existing hardware limitations and/or system requirements. If the frequency band is known the truncated model used should include this information. The following is a modification of the preceding section for those purposes. Using the definition of the discrete Fourier transform[9] (DFT) (included here for completeness), one has

$$\mathrm{DFT}\{x(i)\} = x_d(k) = \Delta T \sum_{\tau=0}^{p-1} x(\tau)Z(k)^{-\tau} \tag{14}$$

and the corresponding inverse transform is

$$x(i) = \Delta f \sum_{\tau=0}^{p-1} x_d(\tau)Z(i)^\tau \tag{15}$$

where ΔT is the sample time, $\Delta f = 1/(p\Delta T)$, and $Z(k) = \exp(j2\pi k/p)$ (the subscript d denotes transform and $j = \sqrt{-1}$). Using these definitions, one can write the truncated output in the following form:

$$\bar{Y}_t(i) = \Delta f \sum_{\tau=0}^{p-1} \bar{Y}_{t,d}(\tau)Z(i)^\tau \tag{16}$$

Substitution of Eq. (16) into Eq. (8) yields

$$J = \Delta f^2 \sum_{k=0}^{p-1} \mathrm{tr}\left\{\sum_{l=0}^{p-1}\sum_{\tau=0}^{p-1} Z(k)^{\tau-l}\bar{Y}_{t,d}^*(l)\bar{Y}_{t,d}(\tau)\right\} \tag{17}$$

where ( )* corresponds to the conjugate transpose. By noting

$$\frac{1}{p}\sum_{\tau=0}^{p-1} Z(\tau)^{j-l} = \delta(j-l) \tag{18}$$

the summation in Eq. (17) is no longer a function of the index k, and the simplified expression is

$$J = p\Delta f^2 \mathrm{tr}\left\{\sum_{\tau=0}^{p-1} \bar{Y}_{t,d}^*(\tau)\bar{Y}_{t,d}(\tau)\right\} \tag{19}$$

Equation (19) gives the error criterion in terms of components of the DFT. The summation is over all the spectral components. To write Eq. (19) in terms of the controllability grammian, the expression for the grammian in Eq. (10) must be transformed using the definitions in Eqs. (14) and (15).

First, the transform of the sequence A^l is given by

$$\mathrm{DFT}(A^l) = \Delta T \sum_{\tau=0}^{p-1} A^\tau Z(k)^{-\tau} = \Delta T \Lambda(k) \tag{20}$$

with

$$\Lambda(k) \equiv Z(k)[Z(k)I - A]^{-1}[I - Z^{-p}(k)A^p] \tag{21}$$

Using Eqs. (20) and (21), the pulse response matrix can be transformed to yield

$$\mathrm{DFT}\{\bar{X}(i)\} = \bar{X}_d(k) = \Delta T \Lambda(k)B \tag{22}$$

and its inverse transform is

$$\bar{X}(k) = \frac{1}{p}\sum_{\tau=0}^{p-1} Z^k(\tau)\Lambda(\tau)B \tag{23}$$

Using Eq. (23) in Eq. (10) results in

$$W_c(p) = \frac{1}{p}\sum_{\tau=0}^{p-1} \Lambda(\tau)BB^T\Lambda^*(\tau) \tag{24}$$

which is an expression for the grammian in terms of a summation of spectral components of the system. Using Eq. (24) and (22), an equivalent expression for Eq. (24) is

$$W_c(p) = \frac{1}{p\Delta T^2}\sum_{\tau=0}^{p-1} \bar{X}_d(\tau)\bar{X}_d^*(\tau) \tag{25}$$

Using Eq. (19) and the partition of $W_c(p)$ corresponding to the truncated states, the truncation criterion can be written

$$J = \frac{1}{p\Delta T^2}\mathrm{tr}\left\{\sum_{\tau=0}^{p-1} C_t^T C_t \bar{X}_d(\tau)\bar{X}_d^*(\tau)\right\} = \mathrm{tr}\{C_t^T C_t w_{tt}\} \tag{26}$$

Although the preceding truncation criterion seems identical to that in Eq. (13), the summation is over all of the spectral components instead of sample time points. The question of model reduction for a particular frequency range can now be addressed. Suppose that a frequency range is given; then only the corresponding spectral components in the range should be included in performance index in Eq. (26).

Model Reduction in Balanced Coordinates

In the preceding sections a criterion has been presented to guide the truncation of states that contribute least to the system pulse response. Using this criterion, there is still the question of how to optimize the performance index to obtain minimum truncation error. A very efficient way to look at this problem was initially discussed in Ref. 3. The basic idea is to transform the system using proper similarity transformations to a form that would ease the minimization of the truncation error. Transformation of the system into a balanced form renders the controllability and observability grammians equal and diagonal. A brief discussion of the procedure follows.

Given the discrete controllability and observability grammians as solutions of

$$\begin{aligned} AW_c(p)A^T - W_c(p+1) &= -BB^T \\ A^TW_o(p)A - W_o(p+1) &= -C^TC \end{aligned} \tag{27}$$

the resulting grammians can be decomposed (using Cholesky decomposition) into

$$W_c(p) = PP^T, \qquad W_o(p) = Q^TQ \tag{28}$$

Next, the matrix $H = QP$ is formed. Decomposing H using singular value decomposition gives

$$H = V\Gamma^2U^T \tag{29}$$

where $V^TV = I$, $U^TU = I$, and Γ is a positive definite diagonal matrix. Defining the transformation matrix

$$\begin{aligned} R &= PU\Gamma^{-1} = Q^{-1}V\Gamma \\ R^{-1} &= \Gamma^{-1}V^TQ = \Gamma U^TP^{-1} \end{aligned} \tag{30}$$

the balanced form is obtained as

$$A_b = R^{-1}AR, \qquad B_b = R^{-1}B, \qquad C_b = CR \tag{31}$$

where in this balanced form the observability and controllability grammians can be shown to be $W_{cb} = W_{ob} = \Gamma^2$. This balanced form permits minimization of Eq. (26) by truncating states corresponding to small diagonal elements of the matrix $C_t^*C_t\Gamma^2$. Each diagonal term in this expression gives the penalty associated with truncating that particular state.

Frequency-Weighted Grammians for Discrete Time Systems

In the definition of the truncation criterion in the frequency domain, the discrete Fourier transform is used to express the index and the grammian in terms of a summation over all spectral components. If a particular frequency range is of interest, the grammian computation must be modified to include only the spectral components in the range. The resulting grammians, which are frequency weighted, can then be used to balance the system. In the following, expressions for the frequency-weighted grammians are developed.

Previously the discrete controllability grammian was expressed as

$$W_c(p) = \frac{1}{p}\sum_{\tau=0}^{p-1} \Lambda(\tau)BB^T\Lambda^*(\tau) \tag{32}$$

and similarly the observability grammian is

$$W_o(p) = \frac{1}{p}\sum_{\tau=0}^{p-1} \Lambda^*(\tau)C^*C\Lambda(\tau) \tag{33}$$

where $\Lambda(i)$ is defined in Eq. (21). The summation is over all of the frequencies up to the Nyquist frequency $f_n = 1/(2\Delta T)$. The preceding expressions converge to a steady-state value in the limit as $p \to \infty$. The limit exists provided that the eigenvalues of A are all within the unit circle, i.e., the system is asymptotically stable. Taking the limit of Eqs. (32) and (33) and dropping terms corresponding to A^p, the steady-state grammians are

$$\begin{aligned} W_c(\Omega) &= \lim_{p\to\infty}\left\{\frac{1}{p}\sum_{l=0}^{p-1} \Psi(l)BB^*\Psi^*(l)\right\} \\ W_o(\Omega) &= \lim_{p\to\infty}\left\{\frac{1}{p}\sum_{l=0}^{p-1} \Psi^*(l)C^*C\Psi(l)\right\} \end{aligned} \tag{34}$$

where the argument Ω, although not needed for the moment, is there to suggest a particular frequency range and $\Psi(l) = [Z(l)I - A]^{-1}$. The steady-state discrete Lyapunov equations can be converted with the aid of the matrix $\Delta(l) = [Z(l)I + A]$ to

$$\begin{aligned} W_c\Delta^*(k)\Psi^*(k) + \Psi(k)\Delta(k)W_c &= 2\Psi(k)BB^T\Psi^*(k) \\ W_o\Delta(k)\Psi(k) + \Psi^*(k)\Delta^*(k)W_o &= 2\Psi^*(k)C^TC\Psi(k) \end{aligned} \tag{35}$$

The preceding equations are satisfied for all values of k. Substitution of Eq. (35) into Eq. (34) yields

$$\begin{aligned} W_c(\Omega) &= W_c\phi^*(\Omega) + \phi(\Omega)W_c \\ W_o(\Omega) &= W_o\phi(\Omega) + \phi^*(\Omega)W_o \end{aligned} \tag{36}$$

where the matrix $\phi(\Omega)$ is defined by

$$\phi(\Omega) = \lim_{p\to\infty}\left\{\frac{1}{2p}\sum_{\tau=0}^{p-1} \Psi(\tau)\Delta(\tau)\right\} \tag{37}$$

One may recall the definition for $Z(l) = \exp(j2\pi l/p)$. Now define $\theta^l = 2\pi l/p$ and $\Delta\theta = 2\pi/p$, and then substitute the result into Eq. (37) to get

$$\phi(\Omega) = \lim_{\Delta\theta\to 0}\frac{1}{4\pi}\sum_{\tau=0}^{\infty} (e^{j\theta_\tau}I - A)^{-1}(e^{j\theta_\tau}I + A)\Delta\theta \tag{38}$$

The preceding equation is the definition of the integral

$$\phi(\Omega) = \frac{1}{4\pi}\int_0^{2\pi} (e^{j\theta}I - A)^{-1}(e^{j\theta}I + A)\,\mathrm{d}\theta \tag{39}$$

Replacing the integration variable using $\xi = e^{j\theta}$, Eq. (39) can be expressed as an integral around a closed contour as follows:

$$\phi(\Omega) = \frac{-j}{4\pi}\oint_{|\xi|=1} (I - A\xi^{-1})^{-1}(I + A\xi^{-1})\xi^{-1}\,\mathrm{d}\xi \tag{40}$$

To evaluate the integral, it is assumed that there are no singularities on the unit circle. This is consistent with the assumption that the system is asymptotically stable. As a verification of Eqs. (40) and (36) the integrand can be expanded in its Laurent series and solved using the residue theorem. Integration around a closed contour yields $\phi(\Omega) = \frac{1}{2}I$. When this result is substituted in Eq. (36), it corresponds to no frequency weighting. To examine the result for Eq. (40) when integrated over a sector, the solution is expressed as

$$\phi(\Omega) = \frac{-j}{4\pi}\{-\log(\xi I) + 2\log(\xi I - A)\}\,\Big|_s \tag{41}$$

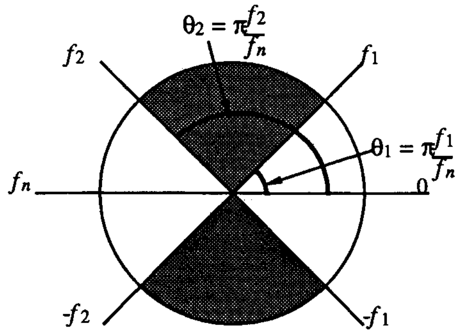

Fig. 1 Integration limits for various frequency ranges.

where s refers to a sector of the unit circle. The sector over which the integration takes place depends on the frequency range of interest. Integration around the unit circle corresponds to a frequency sweep from 0 to f_n .To aid the evaluation of Eq. (41) over a particular range $[f_1, f_2]$, the correspondence between the sector and frequency is shown in Fig. (1).

The top sector is specified by $|\xi| = 1$ and $\theta_1 < \arg(\xi) < \theta_2$. Because of the symmetry of the discrete Fourier transform, integration from π to 2π corresponds to negative frequencies. Therefore, the contribution to the integral for any sector in the upper section has a complex conjugate contribution from the lower sector that should be added.

Once the frequency range of interest is specified, the matrix in Eq. (41) is evaluated and the frequency-weighted grammian is obtained using Eq. (36). Balancing the system in Eq. (1) by using the transformation defined in Eq. (30) produces a diagonal frequency-weighted grammian that, when used in conjunction with Eq. (26), reveals the states to be truncated with the least error.

Numerical Results

To illustrate the model reduction procedure developed, a sixth-order discrete time system is used. The discrete-time system matrices are given by

$$A = \begin{bmatrix} 0.7605 & 0.2394 & 0.0000 & 0.0092 & 0.0008 & 0.0000 \\ 0.0024 & 0.9971 & 0.0005 & 0.0000 & 0.0100 & 0.0000 \\ 0.0000 & 0.0005 & 0.9945 & 0.0000 & 0.0000 & 0.0100 \\ -45.8560 & 45.8286 & 0.0082 & 0.7582 & 0.2417 & 0.0001 \\ 0.4582 & -0.5580 & 0.0994 & 0.0024 & 0.9966 & 0.0010 \\ 0.0002 & 0.0995 & -1.0976 & 0.0000 & 0.0010 & 0.9940 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.0005 & 0.0000 \\ 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 0.0918 & 0.0001 \\ 0.0001 & 0.0010 \\ 0.0010 & 0.0000 \end{bmatrix} \qquad C = \begin{bmatrix} 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 & 0.00 & 0.00 & 0.00 \end{bmatrix} \tag{42}$$

The discrete eigenvalues are $z_i = \{0.7569 \pm 0.6515i,\ 0.9994 \pm 0.0299i,\ 0.9941 \pm 0.1051i\}$. Using a sample time of 0.01 s, the corresponding eigenvalues for the continuous time system are $\lambda_i = \{-0.129 \pm 71.07i,\ -0.020 \pm 2.988i,\ -0.032 \pm 10.53i\}$. The model reduction problem as posed in Eq. (3) is to truncate the states with the smallest contribution to the performance criterion. If Eq. (19) is used, the states truncated are those that contribute the least in a particular frequency band. Assume that the controller design bandwidth is from 0 to 1 Hz. The first step, for model reduction, is to obtain the discrete frequency-weighted grammians in Eq. (36) using Eq. (41). These frequency-weighted grammians are used in Eqs. (21-28) to obtain a balanced system whose frequency-weighted grammians are equal and diagonal. The resulting balanced system matrices are

$$A = \begin{bmatrix} 0.9993 & 0.0299 & -0.0004 & 0.0004 & -0.0001 & 0.0000 \\ -0.0299 & 0.9994 & 0.0004 & -0.0002 & 0.0001 & 0.0000 \\ -0.0004 & -0.0004 & 0.7573 & 0.6515 & -0.0001 & 0.0000 \\ -0.0004 & -0.0002 & -0.6515 & 0.7565 & 0.0003 & -0.0002 \\ 0.0000 & 0.0000 & 0.0001 & 0.0005 & 0.9941 & 0.1051 \\ 0.0000 & 0.0000 & 0.0001 & 0.0003 & -0.1051 & 0.9942 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.0139 & 0.0126 \\ 0.0136 & 0.0124 \\ 0.0303 & -0.0003 \\ 0.0211 & -0.0002 \\ -0.0025 & 0.0003 \\ -0.0023 & 0.0002 \end{bmatrix} \qquad C = \begin{bmatrix} 0.0133 & -0.0130 & 0.0303 & -0.0211 & 0.0018 & -0.0017 \\ 0.0132 & -0.0130 & -0.0003 & 0.0002 & 0.0018 & -0.0016 \end{bmatrix} \tag{43}$$

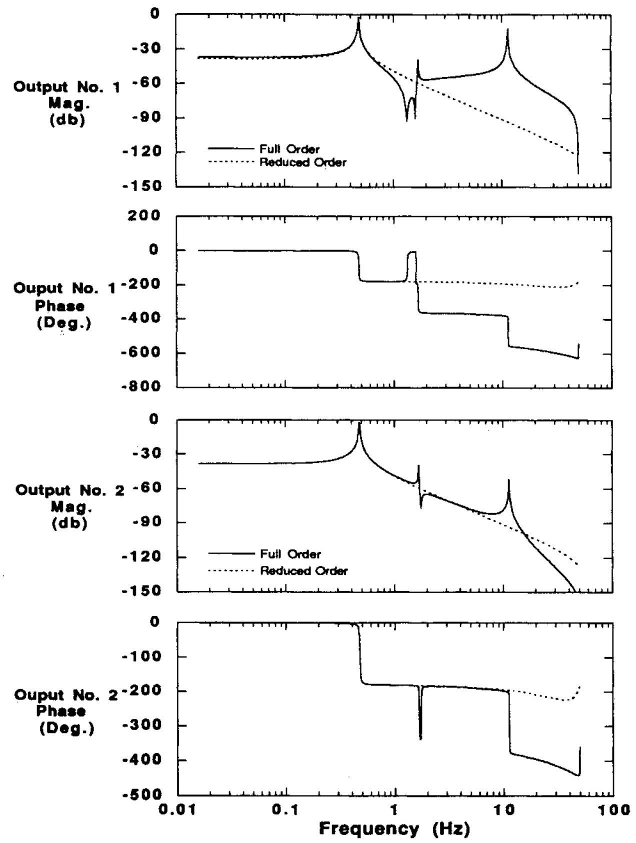

Fig. 2 Comparison of original and reduced-order model frequency response functions using first input; desired range 0–1 Hz.

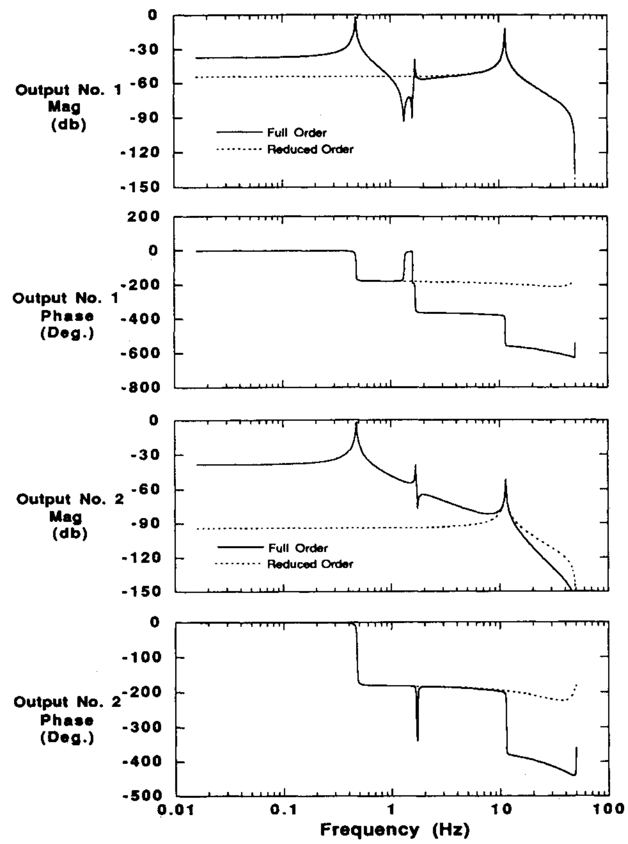

Fig. 3 Comparison of original and reduced-order model frequency response functions using first input; desired range 5–50 Hz.

It is interesting to note that the balanced system is almost block diagonal. Intuitively, a block diagonal form is the most amenable form for truncation because the off-diagonal terms are zero, i.e., truncated states do not affect the retained states. The truncation criterion is $\text{diag}(C_t^T C_t \Gamma^2) = 0.001\{0.91, 0.87, 0.49, 0.23, 0.0002, 0.0001\}$. Since the eigenvalues of the system occur in pairs, the first two states of the system were retained for a performance error of 0.72. The transfer function for the discrete time system is shown in Fig. 2. The dashed line corresponds to the reduced-order model (containing only two states), and the solid line is the original system. Elimination of the highest frequency mode accounts for most of the truncation error. Examining the original transfer matrix, one observes that the truncated mode has a significant contribution to the total response even though it is outside the range of interest.

The second example shows the case when the frequency range of interest is from 5 to 50 Hz. The selection of the range is completely arbitrary but is selected to illustrate the procedure. The system in Eq. (42) is balanced accordingly, and the resulting matrices are

$$A = \begin{bmatrix} 0.7577 & -0.6515 & 0.0001 & -0.0001 & 0.0000 & 0.0000 \\ 0.6515 & 0.7561 & 0.0004 & 0.0000 & 0.0000 & 0.0000 \\ -0.0001 & 0.0004 & 0.7285 & 0.2719 & -0.0102 & -0.0119 \\ -0.0001 & 0.0000 & -0.2719 & 1.2708 & 0.0248 & -0.0209 \\ 0.0000 & 0.0000 & 0.0003 & -0.0006 & 1.0070 & -0.5614 \\ 0.0000 & 0.0000 & 0.0003 & 0.0006 & 0.0208 & 0.9807 \end{bmatrix} \tag{44}$$

$$B = \begin{bmatrix} 0.0303 & -0.0003 \\ -0.0211 & 0.0002 \\ 0.0026 & 0.0026 \\ -0.0034 & -0.0034 \\ 0.0000 & 0.0001 \\ 0.0002 & -0.0002 \end{bmatrix} \qquad C = \begin{bmatrix} 0.0303 & 0.0211 & -0.0026 & -0.0034 & -0.0002 & 0.0001 \\ -0.0003 & -0.0002 & -0.0026 & -0.0034 & -0.0003 & 0.0001 \end{bmatrix}$$

The performance index is given by $\text{diag}(C_t^T C_t \Gamma^2) = 0.001\{0.24, 0.12, 0.00, 0.00, 0.00, 0.00\}$. Retention of the first two states yields an error performance at least two orders of magnitude smaller than the truncated states. Figure 3 shows the corresponding transfer function for the reduced-order model. Excellent matching of the high-frequency magnitude and phase is obtained for both outputs.

Conclusions

The paper presents an extension of a model reduction technique to treat discrete time systems directly. The procedure uses frequency-weighted grammians to determine appropriate balancing transformations. Balancing is used to transform the system into a form amenable for state truncation. Truncation is performed by minimizing the error between the pulse responses from the original and the reduced-order system. Closed-form solutions for the discrete frequency-weighted grammians are developed, and the procedure is demonstrated using a simple sixth-order system. Excellent matching of the transfer function is illustrated for different frequency ranges.

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